ON THE FUNCTIONS OF ORDER STATISTICS FOR RANDOM SAMPLE SIZE

By

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1. Introduction

The study of order statistics has been mostly limited on the determination of functions of order statistics for use in estimation and testing procedures, for a fixed sample size. Various functions studied fully have already been mentioned in David (1970) and have been used to derive non-parametric quick tests. However, there are some situations where the determination of the sample size in advance is not possible, for example in sequential analysis. The main aim of this paper is to develop the theory of order statistics, assuming the sample size as random, for continuous as well as for discrete populations and to illustrate it when the sample size has binomial and Poisson distributions.

2. Distribution of a Single Order Statistic

Let $X_1, X_2, \ldots\ldots\ldots X_n$, be independent and identically distributed random variables with cumulative distribution function (cdf) $F(x)$. Let these variates be arranged in ascending order as

$$X_{(1)} \leqslant X_{(2)} \leqslant \ldots\ldots \leqslant X_{(r)} \leqslant X_{(n)}$$

Let n be not fixed in advance but a discrete random variable with probability mass function (pmf)

$$P(n=i)=p(i) \text{ and cdf } P(s)=\sum_{i=0}^{s} p(i)$$

It is also assumed that n is independent of the distribution of X. Further let $G_r(x)$ be the conditional cdf of r-*th* order statistic $X_{(r)}$ conditioned on the event $\{n \geqslant r\}$.

Then we have

$$G_r(x)=P(X_{(r)}\leqslant x \mid n\geqslant r)$$

(2.1)
$$=\frac{\sum_{i=r}^{\infty} P X_{(r)}\leqslant x/n \mid =i)p(i)}{1-P(r-1)}$$

$$=\frac{\sum_{i=r}^{\infty} p(i)\, I_{F(x)}(r,\, i-r+1)}{1-P(r-1)}$$

where

$$I_a(r,s)=\frac{1}{B(r,s)}\int_0^a u^{r-1}(1-u)^{s-1}\,du,\ 0<a<1$$

$$B(r,s)=\frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)},\ r,s>0$$

For given $F(x)$, $I_{F(x)}(r, i-r+1)$ is tabled by K. Pearson (1934). $p(i)$ being known, $G_r(x)$ may be obtained and percentage points of $X_{(r)}$ may be calculated.

Case 1. For X to be a continuous variate, the pdf of $X_{(r)}$ conditioned on the event $\{n\geqslant r\}$, is given by

(2.2)
$$g_r(x)=\frac{1}{1-P(r-1)}\sum_{i=r}^{\infty} p(i)\ \frac{F^{r-1}(x)\,[1-F(x)]^{i-r}}{B(r,\,i-r+1)}\,f(x)$$

Case 2. For X to be a discrete variate, the conditional pmf of $X_{(r)}$ conditioned on the event $\{n\geqslant r\}$ is

(2.3)
$$g_r(x)=\frac{1}{1-P(r-1)}\sum_{i=r}^{\infty} p(i)$$
$$\Big[\, I_{F(x)}(r,\,i-r+1)-I_{F(x-1)}(r,\,i-r+1)\Big]$$

Expected values of some functions of $X_{(r)}$ can be obtained from (2.2) and (2.3).

Illustration

(*a*) Let n be a binomial variate with pmf

$$p(i)=\binom{N}{i}\pi^{i}(1-\pi)^{N-i},\ i=0,\ 1,\ \ldots,\ N$$

$$0<\pi<1.$$

The conditional cdf of $X_{(r)}$ conditioned on the event $\{n\geqslant r\}$ is given by

$$G_r(x)=\frac{\sum_{i=r}^{N}\binom{N}{i}\pi^{i}(1-\pi)^{N-i}\frac{1}{B(r,i-r+1)}\int_0^{F(x)}u^{r-1}(1-u)^{i-r}du}{\sum_{i=r}^{N}\binom{N}{i}\pi^{i}(1-\pi)^{N-i}}$$

$$=\frac{\frac{N(N-1)\ldots(N-r+1)}{(r-1)!}\int_0^{F(x)}(\pi u)^{r-1}(1-\pi u)^{N-r}\pi du}{I_\pi(r,N-r+1)}$$

$$=\frac{I_{\pi F(x)}(r,\ N-r+1)}{I_\pi(r,\ N-r+1)}$$

For continuous X, the conditional pdf of $X_{(r)}$ conditioned on the event $\{n\geqslant r\}$ is

$$g_r(x)=\frac{\pi^r F^{r-1}(x)[1-\pi F(x)]^{N-r}f(x)}{B(r,\ N-r+1)\,I_\pi(r,\ N-r+1)}$$

For discrete X, the conditional pmf of $X_{(r)}$ conditioned on the event $\{n\geqslant r\}$ is given by

$$g_r(x)=\frac{I_{\pi F(x)}(r,\ N-r+1)-I_{\pi F(x-1)}(r,N-r+1)}{I_\pi(r,\ N-r+1)}$$

(*b*) Let n be a Poisson variate with the probobility mass function

$$p(i)=\frac{e^{-\lambda}\lambda^{i}}{i!},\ i=0,\ 1,\ldots,$$

The conditional cdf of $X_{(r)}$ conditioned on the event $\{x \geqslant r\}$ is

$$G_r(x) = \frac{\sum_{i=r}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!} I_{F(x)}(r, i-r+1)}{\sum_{i=r}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!}}$$

$$= \frac{\lambda^r e^{-\lambda}}{(r-1)!} \left(\sum_{i=r}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!} \right)^{-1} \int_0^{F(x)} u^{r-1} e^{\lambda(1-u)} du$$

$$= \frac{P_{\lambda F(x)}(r)}{P_{\lambda}(r)}$$

where

$$P_a(r) = \sum_{i=r}^{\infty} \frac{e^{-a} a^i}{i!} = \frac{1}{\Gamma(r)} \int_0^a e^{-u} u^{r-1} du$$

For given a, r, $P_a(r)$ is tabled by K. Pearson (1934).

For continuous X, the conditional pdf of $X_{(r)}$ conditioned on the event $\{n \geqslant r\}$ is

$$g_r(x) = \frac{\lambda^r}{\Gamma(r) P_{\lambda}(r)} e^{-\lambda F(x)} F^{r-1}(x) f(x)$$

For discrete X, the conditional pmf of $X_{(r)}$ conditioned on the event $\{n \geqslant r\}$ is

$$g_r(x) = \frac{P_{\lambda F(x)}(r) - P_{\lambda F(x-1)}(r)}{P_{\lambda}(r)}$$

3. THE JOINT DISTRIBUTION OF $X_{(r)}$ AND $X_{(s)}$

Let $G_{rs}(x, y)$ be the conditional joint cumulative distribution function of $X_{(r)}$ and $X_{(s)}$ $\{s \geqslant r\}$ conditioned on the event $\{n \geqslant s > r\}$. Thus for $y \geqslant x$

$G_{rs}(x, y) = P$ (at least r $X_i \leqslant x$, exactly j X_i with $x < X_i \leqslant y / n \geqslant s$)

$$= \frac{1}{\sum_{i=s}^{\infty} p(i)} \sum_{i=s}^{\infty} \sum_{k=r}^{i} \sum_{l=\max(o,\, s-k)}^{i-k} C(i, k, l)\, F^k(x) [F(y) - F(x)]^l \tag{3.1}$$

$$[1 - F(y)]^{i-k-l} p(i)$$

$$=\frac{1}{1-P(s-1)}\sum_{i=s}^{\infty}\frac{p(i)}{B(r,\ s-r,\ i-s+1)}$$

$$\int_{0}^{F(x)}\int_{w}^{F(y)} w^{r-1}\,(v-w)^{s-r-1}(1-v)^{i-s}\,dv\,dw$$

where

$$C(i,k,l)=\binom{i}{k,\ l,\ i-k-l}=\frac{i!}{k!\ l!\ (i-k-l)!}$$

and

$$B(r,\ s-r,\ i-s+1)=\frac{\Gamma(r)\Gamma(s-r)\Gamma(i-s+1)}{\Gamma(i+1)},\ r,\ s-r,i-s+1>0$$

This result could be extended for the joint distribution of k order statistics.

Case 1. For X to be continuous, the conditional joint pdf of $X_{(r)}$ and $X_{(s)}$ conditioned on the event $\{n\geqslant s\}$ is given by

$$(3.2)\quad g_{rs}\,(x,y)=\frac{1}{1-P(s-1)}\sum_{i=s}^{\infty}\frac{p(i)}{B(r,\ s-r,\ i-s+1}$$

$$F^{r-1}(x)[F(y)-F(x)]^{s-r-1}.\,[1-F(y)]^{i-s}f(x)f(y)$$

Case 2. For X to be discrete, the joint pmf of $X_{(r)}$ and $X_{(s)}$ is given as

$$(3.3)\quad g_{rs}(x,\ y)=\frac{1}{1-P(s-1)}\sum_{i=s}^{\infty}\frac{p(i)}{B(r,\ s-r,\ i-s+1)}$$

$$\int_{F(x-1)}^{F(x)}\int_{F(y-1)}^{F(y)} w^{r-1}(v-w)^{s-r-1}.\,(1-v)^{i-s}dv\ dw$$

Note. The other well known results for order statistics can be easily generalised in this case also.

Summary

The basic distribution theory of order statistics and its various functions for continuous and discrete populations, assuming the sample size as random variate and independent of X, is extended. Some of the obtained results are illustrated assuming the sample size has binomial and Poisson distributions.

References

[1] David, H. A. (1970) : Order statistic. John Wiley and Sons Inc.

[2] Pearson, K. (1934) ; Tables of Incomplete B-function. Cambridge University Press.

[3] Pearson, K. (1934) : Tables of Incomplete Gamma Function. Cambridge University Press.